

# Self-adjoint extension approach for spin-1/2 Aharonov-Bohm systems in conical space

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Using two different approaches known in the literature, both based on self-adjoint extension method, we propose a general approach, which is based on the conditions imposed by the physical problem. One of the advantages is that this approach yields the self-adjoint extension parameter in terms of physics of the problem. We apply it for the spin-1/2 Aharonov-Bohm problem in conical space in the nonrelativistic limit. The self-adjoint extension parameter to the bound state and scattering scenarios are determined. Our proposal solves the problem of the arbitrariness of the self-adjoint extension parameter, proposed in Ref. [Phys. Rev. D **40**, 1346 (1989)]. The present method is general and suitable for addressing any quantum system with a singular Hamiltonian that possesses bound and scattering states. As an application, we apply it for the spin-1/2 Aharonov-Bohm problem plus a two-dimensional isotropic harmonic oscillator.

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## I. INTRODUCTION

The Aharonov-Bohm (AB) effect [1] is one of most weird results of quantum phenomena: a charged particle feels effects due non accessible magnetic field only travelling at a magnetic force-free region. The charged particle wave function then acquires a phase shift which would yield produce measurable interference effects. This occurs because the magnetic potential is not zero in the force-free region. Recently it was reported that the AB effect is also realized for neutral particles like photons [2]. In the AB effect of spin-1/2 particles [3, 4], besides the interaction with the magnetic potential, an additional two dimensional  $\delta$  function appears as the mathematical description of the Zeeman interaction between the spin and the magnetic flux tube [5, 6]. This interaction is the basis of the spin-orbit coupling, which causes a splitting on the energy spectrum of atoms depending on the spin state. Hagen [7] argued that this  $\delta$  function contribution to the potential can not be neglected when the system has spin, having shown that changes in the amplitude and scattering cross section are implied in this case.

Point interactions usually appear in quantum systems in the presence of topological defects and have been of great interest in various branches of the physics for its relevance as solvable models [8]. A simple but non trivial example is the case of a cone arising from an effective geometry immersed in several physical systems, such as cosmic strings [9–14], defects in elastic media [15], defects in liquid crystals [16], and so on. In such systems, although the particle does not have access to the core

(defect) region, its wave function and energy spectrum are truly influenced by it.

In quantum mechanics, singularities and pathological potentials are in general dealt with some kind of regularization procedure. A common approach to ensure that the wave function in the presence of a singularity is square-integrable (and therefore might be associated to a bound state) is to force it to vanish on the singularity. More appropriately, an analysis based on the self-adjoint extension method [17] encloses the boundary condition possibilities that still give bound states. The physics of the problem determines which of these possibilities is the right one, leaving no ambiguities [3, 18]. This method has been applied by several authors, in particular, for AB-like systems [3, 6, 19–22]. However, these works provide the most important results (e.g., energy spectrum, phase shift, S matrix) in terms of an arbitrary real parameter, the so called self-adjoint extension parameter.

The self-adjoint extension of symmetric operators is a very powerful mathematical method and it can be applied to various systems in relativistic and nonrelativistic quantum mechanics, supersymmetric quantum mechanics and vortex-like models. The AB interaction within the framework of  $\kappa$  deformed Pauli equation was studied in Ref. [23]. The structure of additional electromagnetic fields to the AB field, for which the Schrödinger, Klein-Gordon and Dirac equations can be solved exactly was addressed in [24]. Correa *et al.* [25] analyzed the supersymmetries of a spin-1/2 particle in the context of magnetic vortex and anyons. They found that there are just two self-adjoint extensions for the Hamiltonian that are compatible with the standard  $N = 2$  supersymmetry. Other approaches for the AB problem supersymmetric extension and for the origin of the hidden supersymmetry were addressed in Refs. [26] and [27], respectively. Slobodeniuk *et al.* [28] presented a comparative study of the electronic properties of a two-dimensional elec-

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tron gas in the presence an infinitesimally thin magnetic field solenoid with relativistic dispersion in graphene and quadratic dispersion as in semiconducting heterostructures. The same study was performed for a graphene system in the inhomogeneous magnetic field consisting of the Aharonov-Bohm flux and a constant background field [29]. In Ref. [30] the Aharonov-Bohm-Casher problem was examined for a charged particle describing a circular path in the presence of a Lorentz-violating background nonminimally coupled to the spinor and gauge fields. For other systems where it is possible to apply the regularization method, see [31–33].

In this paper, we present the details of a general physical regularization method recently proposed in Ref. [34]. It is based on the physics of the problem and one of its particularities is that it gives us the self-adjoint extension parameter for both bound and scattering scenarios. The description of the formalism is based on the works of Kay-Studer (KS) [35] and Bulla-Gesztezy (BG) [36], both using the self-adjoint extension method. We also apply the method for the spin-1/2 AB problem plus a two-dimensional isotropic harmonic oscillator.

The paper is organized as follows. In Sec. II we write the Hamiltonian of the spin-1/2 AB problem and derive the equation of motion that governs the dynamics of the particle. In Sec. III we present the KS and BG self-adjoint extension methods used in the formulation of the regularization method proposed here. The KS method has the advantage of yielding the self-adjoint extension parameter in terms of the physics of the problem, but it is not appropriate for dealing with scattering problems; on the other hand, the BG method is suitable to address both bound and scattering scenarios, with the disadvantage of allowing arbitrary self-adjoint extension parameters. Further, we also derive the expressions for the energy bound state, phase shift and the scattering matrix in terms of the physics of the problem. By combining the KS and BG methods, a relation between the self-adjoint extension parameter and the physical parameters of the problem is found. In Sec. IV we apply the method for the spin-1/2 AB problem in a two-dimensional isotropic harmonic oscillator. We derive the expression for the particle energy spectrum and analyze it in the limit case of the vanishing harmonic oscillator potential recasting the result of pure spin-1/2 AB problem in conical space. In Sec. V we present a brief conclusion.

## II. THE EQUATION OF MOTION

The idealized situation of a relativistic quantum particle in the presence of a cosmic string is an example of a gravitational effect of topological origin, where a particle is transported along a closed curve around the cosmic string [37]. This situation corresponds to the gravitational analogue of the electromagnetic AB effect with the cosmic string replacing the flux tube [38–42]. Such effects are of purely topological origin rather than local. The

bound state for the spinless AB effect around a cosmic string was addressed in [43]. The authors observed that the self-adjoint extension of the Hamiltonian of a particle moving around a shielded cosmic string gives rise to a gravitational analogue of the bound state AB effect. Here, our initial proposal is to analyze the spin-1/2 AB problem in the cosmic string spacetime with an internal magnetic field. The cosmic string background is described by the following metric in cylindrical coordinates  $(t, r, \varphi, z)$ :

$$ds^2 = -dt^2 + dr^2 + \alpha^2 r^2 d\varphi^2 + dz^2, \quad (1)$$

with  $-\infty < (t, z) < \infty$ ,  $r \geq 0$  and  $0 \leq \varphi < 2\pi$ . The parameter  $\alpha$  is related to the linear mass density  $\tilde{m}$  of the string by  $\alpha = 1 - 4\tilde{m}$  runs in the interval  $(0, 1]$  and corresponds to a deficit angle  $\gamma = 2\pi(1 - \alpha)$ . The external gravitational field due to a cosmic string may be approximately described by a commonly called conical geometry. Usually, only the case  $\alpha < 1$  is considered in cosmology, since  $\alpha > 1$  corresponds to a negative mass density cosmic string. For  $\alpha = 1$ , the cone turns into a plane. The spacetime produced by a static straight cosmic string can be obtained in the weak-field and low speeds limit (valid for  $\tilde{m} \ll 1$ ) [37]. The above metric has a cone-like singularity at  $r = 0$  and the curvature tensor of this metric, considered as a distribution, is given by

$$R_{12}^{12} = R_1^1 = R_2^2 = 2\pi \left( \frac{1 - \alpha}{\alpha} \right) \delta^2(\mathbf{r}), \quad (2)$$

where  $\delta^2(\mathbf{r})$  is the two-dimensional delta function in flat space [44]. This implies a two-dimensional conical singularity symmetrical in the  $z$ -axis, which characterizes it as a linear defect.

In order to study the dynamics of the particle in a non-flat spacetime, we should include the spin connection in the differential operator and define the respective Dirac matrices in this manifold. The modified Dirac equation in the curved space reads ( $\hbar = c = 1$ )

$$[i\gamma^\mu(\partial_\mu + \Gamma_\mu) - q\gamma^\mu A_\mu - M]\Psi(x) = 0, \quad (3)$$

where  $q$  is the charge,  $M$  is mass of the particle,  $\Psi(x)$  is a four-component spinorial wave function, and  $\Gamma_\mu$  is the spin connection, which is given by

$$\Gamma_\mu = -\frac{1}{4}\gamma^{(a)}\gamma^{(b)}e_{(a)}^\nu e_{(b)\nu\mu}, \quad (4)$$

and  $\gamma^\mu = e_{(a)}^\mu(x)\gamma^{(a)}$  are the  $\gamma$  matrices in the curved spacetime. We take the basis tetrad [45–47],

$$e_{(a)}^\mu(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \varphi & -\sin \varphi / \alpha r & 0 \\ 0 & \sin \varphi & \cos \varphi / \alpha r & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (5)$$

(with  $\alpha = 1$  giving the flat space-time) satisfying the condition

$$e_{(a)}^\mu(x) e_{(b)}^\nu(x) \eta^{(a)(b)} = g^{\mu\nu}, \quad (6)$$

with  $g^{\mu\nu} = \text{diag}(-, +, +, +)$ . For this conical spacetime the spin connection can be expressed by

$$\gamma^\mu \Gamma_\mu = \kappa \gamma^r, \quad (7)$$

where  $\kappa = (1 - \alpha)/2\alpha r$  and

$$\gamma^r = \cos \varphi \gamma^{(1)} + \sin \varphi \gamma^{(2)} = \begin{pmatrix} 0 & \sigma^r \\ -\sigma^r & 0 \end{pmatrix}. \quad (8)$$

Moreover the  $\alpha$  matrices are now written as

$$\alpha^i = e^i_{(a)} \begin{pmatrix} 0 & \sigma^r \\ -\sigma^r & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad (9)$$

where  $\sigma^i = (\sigma^r, \sigma^\varphi, \sigma^z)$  are the Pauli matrices in cylindrical coordinates obtained from the basis tetrad (5).

For the specific tetrad basis used here, the spin connection is

$$\Gamma_\mu = (0, 0, \Gamma_\varphi, 0), \quad (10)$$

with the non-vanishing element given as

$$\Gamma_\varphi = i \frac{(1 - \alpha)}{2} \Sigma^3, \quad (11)$$

with

$$\Sigma^3 = \begin{pmatrix} \sigma^z & 0 \\ 0 & \sigma^z \end{pmatrix}. \quad (12)$$

We are interested in the non-relativistic limit of the Dirac equation, so it is convenient to express it in terms of a Hamiltonian formalism:

$$\hat{H}\Psi(x) = i\partial_t\Psi(x), \quad (13)$$

with

$$\hat{H} = \alpha^i (-i\partial_i + qA_i) + qA_0 - i\gamma^0 \gamma^\mu \Gamma_\mu + \beta M, \quad (14)$$

where we have defined the notation

$$\alpha^i = \gamma^0 \gamma^i, \quad \gamma^0 = \beta,$$

and the Dirac matrices obey the anticommutator relations

$$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}.$$

In the non-relativistic limit we have to extract the temporal behavior of wave function. Writing

$$\Psi = e^{-iMt} \begin{pmatrix} \Phi \\ X \end{pmatrix}, \quad (15)$$

we obtain the equations

$$(\sigma^i \pi^i + i\kappa \sigma^r) X + qA_0 \Phi = i\partial_t \Phi, \quad (16)$$

$$(\sigma^i \pi^i + i\kappa \sigma^r) \Phi + qA_0 X - 2MX = i\partial_t X, \quad (17)$$

where  $\pi^i = -i\partial_i + qA_i$ . In the non-relativistic limit the “small” component of the wave function,  $X$ , can be related with its “large” one,  $\Phi$ , by

$$X = \frac{1}{2M} (\sigma^i \pi^i + i\kappa \sigma^r) \Phi. \quad (18)$$

Now substituting this result in Eq. (16), we find

$$\hat{H}_{NR} \Phi = i\partial_t \Phi, \quad (19)$$

where

$$\hat{H}_{NR} = \frac{1}{2M} (\sigma^i \pi^i + i\kappa \sigma^r) (\sigma^j \pi^j + i\kappa \sigma^r) + qA_0. \quad (20)$$

The magnetic flux tube in the background space described by the metric above considered is related to the magnetic field by

$$s\mathbf{B} = s(\nabla \times \mathbf{A}) = \frac{s\bar{\phi}}{\alpha} \frac{\delta(r)}{r} \hat{\mathbf{z}}, \quad (21)$$

where  $\bar{\phi} = \phi/2\pi$  is the flux parameter, the vector potential in the Coulomb gauge is

$$\mathbf{A}_\varphi = \frac{\bar{\phi}}{\alpha r} \hat{\varphi}, \quad A_0 = 0, \quad (22)$$

and  $s = \pm 1$  is twice the spin projection parameter [7]. The parameter  $s$  implies that the Dirac equation describes the planar motion (in the absence of the  $z$  coordinate) of the particle having only one projection of the three-dimensional spin vector. The choice (21) also gives a flux tube coinciding with the cosmic string and the  $z$ -axis.

Using the relations

$$\sigma^i \sigma^j = g^{ij} + i\epsilon^{ijk} \sigma^k, \quad (23)$$

$$\sigma^i (\partial_i \sigma^j) = \frac{1}{r} g^{ij}, \quad (24)$$

the Hamiltonian (20) can be written as

$$\hat{H}_{NR} = \frac{1}{2M} \left[ \frac{1}{i} \nabla_\alpha - \frac{q\bar{\phi}}{\alpha r} + \frac{1 - \alpha}{2\alpha r} \sigma_z \right]^2 + \frac{\eta}{2M} \frac{\delta(r)}{r}, \quad (25)$$

where

$$\eta = -\frac{qs\bar{\phi}}{\alpha}, \quad (26)$$

and

$$\nabla_\alpha^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{\alpha^2 r^2} \frac{\partial^2}{\partial \varphi^2}, \quad (27)$$

is the Laplacian operator in the conical space. For this system the total angular momentum operator,

$$\hat{J} = -i\partial_\varphi + \frac{1}{2}\sigma^z, \quad (28)$$

Possibilities	$q$	$s$	$\eta$	$\overline{\mathcal{U}}_{\text{short}}$	State
1	$< 0$	$+1$	$> 0$	$> 0$	Scattering
2	$> 0$	$-1$	$> 0$	$> 0$	Scattering
3	$< 0$	$-1$	$< 0$	$< 0$	Bound and Scattering
4	$> 0$	$+1$	$< 0$	$< 0$	Bound and Scattering

TABLE I. Possibilities to occur  $\overline{\mathcal{U}}_{\text{short}} > 0$  and  $\overline{\mathcal{U}}_{\text{short}} < 0$ .

commutes with the effective Hamiltonian. So it is possible to express the eigenfunctions of the two dimensional Hamiltonian in terms of the eigenfunctions of  $\hat{J}$ . These functions have the form

$$\Phi(t, r, \varphi) = e^{-i\mathcal{E}t} \begin{pmatrix} f_{\mathcal{E}}(r)e^{i(m-s/2)\varphi} \\ g_{\mathcal{E}}(r)e^{i(m+s/2)\varphi} \end{pmatrix}, \quad (29)$$

with  $m = n + 1/2$ ,  $n \in \mathbb{Z}$ . At the same time, the radial equation for  $f_{\mathcal{E}}(r)$  becomes

$$\mathcal{H}f_{\mathcal{E}}(r) = \mathcal{E}f_{\mathcal{E}}(r), \quad (30)$$

where

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{U}_{\text{short}}, \quad (31)$$

$$\mathcal{H}_0 = -\frac{1}{2M} \left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{j^2}{r^2} \right], \quad (32)$$

$$\mathcal{U}_{\text{short}} = \frac{\eta}{2M} \frac{\delta(r)}{r}, \quad (33)$$

and

$$j = \frac{1}{\alpha} \left( m - \frac{s}{2} - q\bar{\phi} + \frac{1-\alpha}{2} \right). \quad (34)$$

The Hamiltonian in Eq. (31) governs the quantum dynamics of a spin-1/2 charged particle in the conical space-time, with a magnetic field  $\mathbf{B}$  along the z-axis, i.e., a spin-1/2 AB problem in the conical space. Let us consider a conical defect with a radius  $r_0$  nucleus, so it is suitable to write  $\mathcal{U}_{\text{short}}(r)$  as [5, 7]

$$\overline{\mathcal{U}}_{\text{short}}(r) = \eta \frac{\delta(r - r_0)}{r_0}, \quad (35)$$

and, at the end, the limit  $r_0 \rightarrow 0$  is taken. Although the functional structure of  $\mathcal{U}_{\text{short}}$  and  $\overline{\mathcal{U}}_{\text{short}}$  are quite different, as discussed in [7], we are free to use any form of potential provided that only the contribution of the form (33) is excluded.

Let us now study the  $\overline{\mathcal{U}}_{\text{short}}$  signal, which provides us the possibilities to obtain both scattering or bound states. Therefore, it is important to highlight such possibilities. This can be done by studying the coefficient  $\eta$  as shown in Table I for  $q \leq 0$  and  $s = \pm 1$ . Since we have two possibilities for achieving bound states and scattering, we will focus our attention first on the conditions giving bound states. Afterwards, only when we study the scattering problem is that we will take into account the other two conditions.

### III. SELF-ADJOINT EXTENSIONS

For smooth functions,  $g \in C_0^\infty(\mathbb{R}^2)$  with  $g(0) = 0$ , we should have

$$\mathcal{H}g = \mathcal{H}_0g, \quad (36)$$

and hence it is reasonable to interpret [48–50] the Hamiltonian (31) as a self-adjoint extension of

$$\mathcal{H}_0|_{C_0^\infty(\mathbb{R}^2 \setminus \{0\})}. \quad (37)$$

In order to proceed to the self-adjoint extensions of (32), we decompose the Hilbert space  $\mathfrak{H} = L^2(\mathbb{R}^2)$  with respect to the angular momentum  $\mathfrak{H} = \mathfrak{H}_r \otimes \mathfrak{H}_\varphi$ , where  $\mathfrak{H}_r = L^2(\mathbb{R}^+, r dr)$  and  $\mathfrak{H}_\varphi = L^2(\mathcal{S}^1, d\varphi)$ , with  $\mathcal{S}^1$  denoting the unit sphere in  $\mathbb{R}^2$ . The operator  $-\partial_\varphi^2$  is essentially self-adjoint in  $L^2(\mathcal{S}^1, d\varphi)$  [17] and we obtain the operator  $\mathcal{H}_0$  in each angular momentum sector. Now, using the unitary operator  $V : L^2(\mathbb{R}^+, r dr) \rightarrow L^2(\mathbb{R}^+, dr)$ , given by  $(Vg)(r) = r^{1/2}g(r)$ , the operator  $\mathcal{H}_0$  becomes

$$h_0 = V\mathcal{H}_0V^{-1} = -\frac{1}{2M} \left[ \frac{d^2}{dr^2} + \left( j^2 - \frac{1}{4} \right) \frac{1}{r^2} \right], \quad (38)$$

which is essentially self-adjoint for  $|j| \geq 1$ , while for  $|j| < 1$  it admits a one-parameter family of self-adjoint extensions [17]. To characterize this family, we will use the KS [35] and the BG [36] approaches, both based in boundary conditions.

In the KS approach, the boundary condition is a match of the logarithmic derivatives of the zero-energy solutions for Eq. (30) and the solutions for the problem  $\mathcal{H}_0$  plus self-adjoint extension. In the BG approach, the boundary condition is a mathematical limit allowing divergent solutions for the Hamiltonian (32) at isolated points, provided they remain square integrable.

#### A. KS method

Now, the goal is to find the bound states for the Hamiltonian (31). Following [35], we temporarily forget the  $\delta$  function potential and find the boundary conditions allowed for  $\mathcal{H}_0$ . But the self-adjoint extension provides an infinity of possible boundary conditions, so that it can not give us the true physics of the problem. Nevertheless, once the physics at  $r = 0$  is known [3, 51, 52], it is possible to determine any arbitrary parameter coming from the self-adjoint extension, in such a way it is possible to obtain a complete description of the problem. Since we have a singular point, we must guarantee that the Hamiltonian is self-adjoint in the region of motion. Note that even if  $\mathcal{H}_0^\dagger = \mathcal{H}_0$ , their domains could be different. The operator  $\mathcal{H}_0$ , with domain  $\mathcal{D}(\mathcal{H}_0)$ , is self-adjoint if  $\mathcal{D}(\mathcal{H}_0^\dagger) = \mathcal{D}(\mathcal{H}_0)$  and  $\mathcal{H}_0^\dagger = \mathcal{H}_0$ . We must find the deficiency subspaces,  $\mathcal{N}_\pm$ , which are defined by

[17]

$$\begin{aligned}\mathcal{N}_+ &= \left\{ \psi \in \mathcal{D}(\mathcal{H}_0^\dagger), \mathcal{H}_0^\dagger \psi = z_+ \psi, \Im z_+ > 0 \right\}, \\ \mathcal{N}_- &= \left\{ \psi \in \mathcal{D}(\mathcal{H}_0^\dagger), \mathcal{H}_0^\dagger \psi = z_- \psi, \Im z_- < 0 \right\},\end{aligned}$$

with dimensions  $n_+$  and  $n_-$ , respectively, which are called deficiency indices of  $\mathcal{H}_0$  [17]. A necessary and sufficient condition for  $\mathcal{H}_0$  being essentially self-adjoint is that  $n_+ = n_- = 0$ . On the other hand, if  $n_+ = n_- \geq 1$  the operator  $\mathcal{H}_0$  has an infinite number of self-adjoint extensions parametrized by the unitary operators  $U : \mathcal{N}_+ \rightarrow \mathcal{N}_-$ .

Next we substitute the problem in Eq. (30) by the eigenvalue equation,

$$\mathcal{H}_0 f_{\varrho, \varepsilon} = \varepsilon f_{\varrho, \varepsilon}, \quad (39)$$

with  $f_{\varrho, \varepsilon}$  labeled by a parameter  $\varrho$  which is related to the behavior of the wave function in the limit  $r \rightarrow r_0$ . But for imposing any boundary condition (e.g.  $f = 0$  at  $r = 0$ ) it is necessary to discover what boundary conditions are allowed to  $\mathcal{H}_0$ .

Now, in order to find the full domain of  $\mathcal{H}_0$  in  $L^2(\mathbb{R}^+, r dr)$ , we have to find its deficiency subspace. To do this, we solve the eigenvalue equation

$$\mathcal{H}_0^\dagger f_\pm = \pm i k_0 f_\pm, \quad (40)$$

where  $k_0 \in \mathbb{R}$  is introduced for dimensional reasons. Since  $\mathcal{H}_0^\dagger = \mathcal{H}_0$ , the only square integrable functions which are solutions of Eq. (40) are the modified Bessel functions,

$$f_\pm = K_j(r\sqrt{\mp\varepsilon}), \quad (41)$$

with  $\varepsilon = 2iMk_0$ . These functions are square integrable only in the range  $j \in (-1, 1)$ , for which  $\mathcal{H}_0$  is not self-adjoint. The dimension of such deficiency subspace is  $(n_+, n_-) = (1, 1)$ . So we have two situations for  $j$ , i.e.,

$$\begin{aligned}-1 &< j < 0, \\ 0 &< j < 1.\end{aligned} \quad (42)$$

To treat both cases of Eq. (42) simultaneously, it is more convenient to use

$$f^\pm = K_{|j|}(r\sqrt{\mp\varepsilon}). \quad (43)$$

Therefore, according to the von Neumann-Krein theory, the domain of  $\mathcal{H}_0^\dagger$  is given by

$$\mathcal{D}(\mathcal{H}_0^\dagger) = \mathcal{D}(\mathcal{H}_0) \oplus \mathcal{N}_+ \oplus \mathcal{N}_-. \quad (44)$$

The self-adjoint extension approach consists, essentially, in extending the domain  $\mathcal{D}(\mathcal{H}_0)$  to match  $\mathcal{D}(\mathcal{H}_0^\dagger)$  and therefore turning  $\mathcal{H}_0$  self-adjoint. We then get

$$\mathcal{D}(\mathcal{H}_{0, \varrho}) = \mathcal{D}(\mathcal{H}_0^\dagger) = \mathcal{D}(\mathcal{H}_0) \oplus \mathcal{N}_+ \oplus \mathcal{N}_-, \quad (45)$$

For each  $\varrho$ , we have a possible domain for  $\mathcal{H}_0$ . But it will be the physical situation the factor that will determine

the value of  $\varrho$ . Thus,  $\mathcal{D}(\mathcal{H}_{0, \varrho})$  in  $L^2(\mathbb{R}^+, r dr)$  is given by the set of functions [17]

$$f_{\varrho, \varepsilon}(r) = f_{|j|}(r) + C [K_{|j|}(r\sqrt{-\varepsilon}) + e^{i\varrho} K_{|j|}(r\sqrt{\varepsilon})], \quad (46)$$

where  $f_{|j|}(r)$ , with  $f_{|j|}(r_0) = \dot{f}_{|j|}(r_0) = 0$  ( $\dot{f} \equiv df/dr$ ), is the regular wave function when we do not have  $\overline{\mathcal{U}}_{\text{short}}(r)$ . The last term in Eq. (46) gives the correct behavior for the wave function when  $r = r_0$ . The parameter  $\varrho \in [0, \pi)$  represents a choice for the boundary condition. As we shall see below, the physics of the problem determines such parameter without ambiguity. In fact,  $\varrho$  describes the coupling between  $\overline{\mathcal{U}}_{\text{short}}(r)$  and the wave function. Thus, it must be expressed in terms of  $\alpha$ , the defect core radius  $r_0$  and the effective angular momentum  $j$ . The next step is to find a fitting for  $\varrho$  compatible with  $\overline{\mathcal{U}}_{\text{short}}(r)$ . In this sense, we write Eqs. (30) and (39) for  $\varepsilon = 0$ ,

$$\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{j^2}{r^2} + \overline{\mathcal{U}}_{\text{short}} \right] f_0 = 0, \quad (47)$$

$$\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{j^2}{r^2} + \right] f_{\rho, 0} = 0, \quad (48)$$

implying the zero-energy solutions  $f_0$  and  $f_{\rho, 0}$ . Now, we require the continuity for the logarithmic derivative

$$r_0 \frac{\dot{f}_0}{f_0} \Big|_{r=r_0} = r_0 \frac{\dot{f}_{\rho, 0}}{f_{\rho, 0}} \Big|_{r=r_0} \quad (49)$$

The left-hand side of this equation can be achieved integrating (47) from 0 to  $r_0$ ,

$$\begin{aligned} \int_0^{r_0} \frac{1}{r} \frac{d}{dr} \left( r \frac{df_0(r)}{dr} \right) r dr &= \eta \int_0^{r_0} f_0(r) \frac{\delta(r-r_0)}{r} r dr \\ &+ j^2 \int_0^{r_0} \frac{f_0(r)}{r^2} r dr. \end{aligned} \quad (50)$$

Now, considering that  $f_0(r)/r^2$  does not change a lot in  $[0, r_0]$ , we find

$$\int_0^{r_0} \frac{f_0(r)}{r^2} r dr \approx \frac{f_0(r_0)}{r_0^2} \int_0^{r_0} r dr = \frac{f_0(r_0)}{2}. \quad (51)$$

So, we have

$$r_0 \frac{\dot{f}_0}{f_0} \Big|_{r=r_0} = \eta + \frac{j^2}{2}. \quad (52)$$

Since  $r_0 \rightarrow 0$ , the right-hand side of Eq. (49) is calculated using the asymptotic representation for  $K_{|j|}(x)$  in the limit  $x \rightarrow 0$ , given by

$$K_{|j|}(x) \sim C_j \left[ \frac{x^{-|j|}}{2^{-|j|}\Gamma(1-|j|)} - \frac{x^{|j|}}{2^{|j|}\Gamma(1+|j|)} \right], \quad (53)$$

where  $C_j = \pi/2 \sin(\pi|j|)$ . Here a comment is in order upon Eq. (53). In Ref. [34], this expression has a sign



mistake. Here, we present the correct expression and the respective implications on the results [53]. Thus, taking into account (46), we arrive at

$$r_0 \frac{\dot{f}_{\rho,0}}{f_{\rho,0}} \Big|_{r=r_0} = \frac{\dot{\Omega}_\varrho(r)}{\Omega_\varrho(r)} \Big|_{r=r_0}, \quad (54)$$

where

$$\Omega_\varrho(r) = \left[ \frac{(r\sqrt{-\varepsilon})^{-|j|}}{2^{-|j|}\Gamma(1-|j|)} - \frac{(r\sqrt{-\varepsilon})^{|j|}}{2^{|j|}\Gamma(1+|j|)} \right] + e^{i\varrho} \left[ \frac{(r\sqrt{\varepsilon})^{-|j|}}{2^{-|j|}\Gamma(1-|j|)} - \frac{(r\sqrt{\varepsilon})^{|j|}}{2^{|j|}\Gamma(1+|j|)} \right]. \quad (55)$$

Inserting (52) and (54) in (49) we obtain

$$\frac{\dot{\Omega}_\varrho(r)}{\Omega_\varrho(r)} \Big|_{r=r_0} = \eta + \frac{j^2}{2}, \quad (56)$$

which gives us the parameter  $\varrho$  in terms of the physics of the problem, that is, the correct behavior of the wave functions when  $r \rightarrow r_0$ , or the coupling between the short-ranged potential  $\bar{U}_{\text{short}}(r)$  and the wave function.

Next we will find the bound states of the Hamiltonian  $\mathcal{H}_0$  and using (56) the spectrum of  $\mathcal{H}$  will be determined without any arbitrary parameter. Then, from Eq. (39) we achieve the modified Bessel equation ( $\kappa^2 = -2M\mathcal{E}$ ,  $\mathcal{E} < 0$ )

$$\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \left( \frac{j^2}{r^2} + \kappa^2 \right) \right] f_{\varrho,\mathcal{E}}(r) = 0, \quad (57)$$

whose general solution is given by

$$f_{\varrho,\mathcal{E}}(r) = K_{|j|} \left( r\sqrt{-2M\mathcal{E}} \right). \quad (58)$$

Since these solutions belong to  $\mathcal{D}(\mathcal{H}_{0,\varrho})$ , it is of the form (46), that is,

$$f_{\varrho,\mathcal{E}}(r) = f_{|j|}(r) + C \left[ K_{|j|}(r\sqrt{-\varepsilon}) + e^{i\varrho} K_{|j|}(r\sqrt{\varepsilon}) \right], \quad (59)$$

for some  $\varrho$  selected from the physics of the problem. So, we substitute (58) into (59) and compute  $r_0 f_{\varrho,\mathcal{E}} / f_{\varrho,\mathcal{E}}|_{r=r_0}$  using (53). After a straightforward calculation we have the relation

$$\begin{aligned} r_0 \frac{\dot{f}_{\varrho,\mathcal{E}}(r)}{f_{\varrho,\mathcal{E}}(r)} \Big|_{r=r_0} &= \frac{|j| \left[ r_0^{2|j|} \Gamma(-|j|)(-M\mathcal{E})^{|j|} - 2^{|j|} \Gamma(|j|) \right]}{r_0^{2|j|} \Gamma(-|j|)(-M\mathcal{E})^{|j|} + 2^{|j|} \Gamma(|j|)} \\ &= \frac{\dot{\Omega}_\varrho(r)}{\Omega_\varrho(r)} \Big|_{r=r_0}. \end{aligned} \quad (60)$$

By using Eq. (56) and solving the above equation for  $\mathcal{E}$ , we find the sought energy spectrum

$$\mathcal{E} = -\frac{2}{Mr_0^2} \left[ \left( \frac{j^2 - 2|j| + 2\eta}{j^2 + 2|j| + 2\eta} \right) \frac{\Gamma(1+|j|)}{\Gamma(1-|j|)} \right]^{1/|j|}. \quad (61)$$

Notice that there is no arbitrary parameters in the above equation.

## B. BG method

The approach used in the previous sections give us the energy spectrum in terms of the physics of the problem, but is not appropriate for dealing with scattering problems. Furthermore, it selects the value for the parameter  $\varrho$ . On the other hand, the approach in [36] is suitable to address both bound and scattering scenarios, with the disadvantage of allowing arbitrary self-adjoint extension parameters. By comparing the results of these two approaches for bound states, the self-adjoint extension parameter can be determined in terms of the physics of the problem. Here, all self-adjoint extensions  $\mathcal{H}_{0,\lambda_j}$  of  $\mathcal{H}_0$  are parametrized by the boundary condition at the origin,

$$g_0(r) = \lambda_j \lim_{r \rightarrow 0^+} \frac{1}{r^{|j|}} \left[ g(r) - g_0(r') \frac{1}{r^{|j|}} \right]. \quad (62)$$

with

$$g_0(r) = \lim_{r \rightarrow 0^+} r^{|j|} g(r). \quad (63)$$

where  $\lambda_j$  is the self-adjoint extension parameter. In [8] it was shown that there is a relation between the self-adjoint extension parameter  $\lambda_j$  and the parameter  $\varrho$  used in the previous sections. The parameter  $\varrho$ , is associated with the mapping of deficiency subspaces and extend the domain of operator to make it self-adjoint, being a mathematical parameter. The self-adjoint extension parameter  $\lambda_j$  have a physical interpretation, it represents the scattering length [54] of  $\mathcal{H}_{0,\lambda_j}$  [8]. For  $\lambda_j = 0$  we have the free Hamiltonian (without the  $\delta$  function) with regular wave functions at origin and for  $\lambda_j \neq 0$  the boundary condition in (62) permit a  $r^{-|j|}$  singularity in the wave functions at origin.

### 1. Bound states

The solutions for

$$\mathcal{H}_0 f_\mathcal{E} = k^2 f_\mathcal{E}, \quad (64)$$

with  $k^2 = 2M\mathcal{E}$  for  $r \neq 0$ , taking into account both cases in (42) simultaneously, can be written as ( $\rho = 2ikr$ )

$$\begin{aligned} f_\mathcal{E}(r) &= A_j e^{-\frac{\rho}{2}} \rho^{|j|} M\left(\frac{1}{2} + |j|, 1 + 2|j|, \rho\right) \\ &+ B_j e^{-\frac{\rho}{2}} \rho^{-|j|} M\left(\frac{1}{2} - |j|, 1 - 2|j|, \rho\right), \end{aligned} \quad (65)$$

where  $M(a, b, z)$  represents the confluent hypergeometric function of the first kind, and  $A_j, B_j$  are the coefficients of the regular and irregular solutions, respectively. By implementing Eq. (65) into the boundary condition (62), we derive the following relation between the coefficients  $A_j$  and  $B_j$ :

$$\lambda_j A_j = (2ik)^{-2|j|} B_j \left( 1 + \frac{\lambda_j k^2}{4(1-|j|)} \lim_{r \rightarrow 0^+} r^{2-2|j|} \right). \quad (66)$$

In the above equation, the coefficient of  $B_j$  diverges as  $\lim_{r \rightarrow 0^+} r^{2-2|j|}$ , if  $|j| > 1$ . Thus,  $B_j$  must be zero for  $|j| > 1$ , and the condition for the occurrence of a singular solution is  $|j| < 1$ . So, the presence of an irregular solution stems from the fact the operator is not self-adjoint for  $|j| < 1$ , and this irregular solution is associated with a self-adjoint extension of the operator  $\mathcal{H}_0$  [55, 56]. In other words, the self-adjoint extension essentially consists in including irregular solutions in  $\mathcal{D}(\mathcal{H}_0)$ , which allows to select an appropriate boundary condition for the problem.

In the present system the energy of a bound state has to be negative, so that  $k$  is a pure imaginary,  $k = i\kappa$ . With the substitution  $k \rightarrow i\kappa$ , we have ( $\rho' = -2\kappa r$ )

$$f_{\mathcal{E}}^{\mathcal{B}}(r) = A_j e^{-\frac{\rho'}{2}} \rho'^{|j|} M\left(\frac{1}{2} + |j|, 1 + 2|j|, \rho'\right) + B_j e^{-\frac{\rho'}{2}} \rho'^{-|j|} M\left(\frac{1}{2} - |j|, 1 - 2|j|, \rho'\right). \quad (67)$$

For Eq. (67) representing a bound state, the solution  $f_{\mathcal{E}}^{\mathcal{B}}(r)$  must vanish for  $r \rightarrow \infty$ , i.e., it must be normalizable. By using the asymptotic representation of  $M(a, b, z)$  for  $r \rightarrow \infty$ ,

$$M(a, b, z) \sim \frac{\Gamma(b)}{\Gamma(a)} e^z z^{a-b} + \frac{\Gamma(b)}{\Gamma(b-a)} (-z)^{-a}, \quad (68)$$

where  $\Gamma(z)$  is the Euler  $\Gamma$  function, the normalizability condition yields the relation

$$A_j \frac{\Gamma(1+2|j|)}{\Gamma(\frac{1}{2}+|j|)} + B_j \frac{\Gamma(1-2|j|)}{\Gamma(\frac{1}{2}-|j|)} = 0. \quad (69)$$

Or, using the well-known relations for the  $\Gamma$  function

$$\Gamma(z+1) = z\Gamma(z), \quad \Gamma(2z) = (2\pi)^{-\frac{1}{2}} 2^{2z-\frac{1}{2}} \Gamma(z)\Gamma(z+\frac{1}{2}), \quad (70)$$

we can write

$$B_j = -16^{|j|} \frac{\Gamma(1+|j|)}{\Gamma(1-|j|)} A_j. \quad (71)$$

From Eq. (66), for  $|j| < 1$  we have

$$B_j = \lambda_j (-2\kappa)^{2|j|} A_j, \quad (72)$$

and by using Eq. (71), the bound state energy is

$$\mathcal{E} = -\frac{2}{M} \left[ -\frac{1}{\lambda_j} \frac{\Gamma(1+|j|)}{\Gamma(1-|j|)} \right]^{1/|j|}. \quad (73)$$

This coincides with Eq. (3.13) of Ref. [20] for  $\alpha = 1$ , i.e., the spin-1/2 AB problem in Euclidean space. By comparing Eq. (73) with Eq. (61), we find

$$\frac{1}{\lambda_j} = -\frac{1}{r_0^{2|j|}} \left( \frac{j^2 - 2|j| + 2\eta}{j^2 + 2|j| + 2\eta} \right). \quad (74)$$

We have thus attained a relation between the self-adjoint extension parameter and the physical parameters of the

problem,  $j$  and  $r_0$ . It should be mentioned that some relations involving the self-adjoint extension parameter and the  $\delta$  function coupling constant were previously obtained by using Green's function in Ref. [21] and the renormalization technique in Ref. [19], being both, however, deprived from a clear physical interpretation.

The (unnormalized) bound state wave function is given by

$$f_{\mathcal{E}}^{\mathcal{B}}(r, \varphi) = \frac{2^{2|j|}}{\Gamma(-|j|)} K_{|j|}(-\sqrt{-2M\mathcal{E}} r) e^{i(m-s/2)\varphi}, \quad (75)$$

where  $K_{|j|}(z)$  is the modified Bessel function of the second kind.

## 2. Scattering

Once the bound energy problem has been examined, let us now analyze the AB scattering scenario. In this case, the boundary condition is again given by Eq. (62), but with the replacement  $\lambda_j \rightarrow \lambda_j^s$ , where  $\lambda_j^s$  is the self-adjoint extension parameter for the scattering problem. In the scattering analysis it is more convenient to write the solution for Eq. (64) in terms of Bessel functions

$$f_{\mathcal{E}}(r) = C_j J(|j|, kr) + D_j Y(|j|, kr), \quad (76)$$

with  $C_j$  and  $D_j$  being constants. Upon replacing  $f_{\mathcal{E}}(r)$  in the boundary condition (62), we obtain

$$\lambda_j^s C_j \xi k^{|j|} = D_j [\zeta k^{-|j|} - \lambda_j^s (\beta k^{|j|} + \zeta \nu k^{-|j|} \lim_{r \rightarrow 0^+} r^{2-2|j|})], \quad (77)$$

where

$$\xi = \frac{1}{2^{|j|} \Gamma(1+|j|)}, \quad \zeta = -\frac{2^{|j|} \Gamma(|j|)}{\pi}, \quad \beta = -\frac{\cos(\pi|j|) \Gamma(-|j|)}{\pi 2^{|j|}}, \quad \nu = \frac{k^2}{4(1-|j|)}. \quad (78)$$

As in the bound state calculation, whenever  $|j| < 1$ , we have  $D_j \neq 0$ ; again, this means that there arises the contribution of the irregular solution  $Y$  at the origin when the operator is not self-adjoint. Thus, for  $|j| < 1$ , we obtain

$$\lambda_j^s C_j \xi k^{|j|} = D_j (\zeta k^{-|j|} - \lambda_j^s \beta k^{|j|}), \quad (79)$$

and by substituting the values of  $j$ ,  $\zeta$  and  $\eta$  into above expression we find

$$D_j = -\mu_j^{\lambda_j^s} C_j, \quad (80)$$

where

$$\mu_j^{\lambda_j^s} = \frac{\lambda_j^s k^{2|j|} \Gamma(1-|j|) \sin(|j|\pi)}{\lambda_j^s k^{2|j|} \Gamma(1-|j|) \cos(\pi|j|) + 4^{|j|} \Gamma(1+|j|)}. \quad (81)$$

Since  $\delta$  is a short range potential, it follows that the behavior of  $f_{|j|}$  for  $r \rightarrow \infty$  is given by [57]

$$f_{|j|}(r) \sim \sqrt{\frac{2}{\pi k r}} \cos \left( k r - \frac{|m|\pi}{2} - \frac{\pi}{4} + \delta_j^{\lambda_j^s}(k, \bar{\phi}) \right), \quad (82)$$

where  $\delta_j^{\lambda_j^s}(k, \bar{\phi})$  is a scattering phase shift. The phase shift is a measure of the argument difference to the asymptotic behavior of the solution  $J(|m|, kr)$  of the radial free equation that is regular at the origin. By using the asymptotic behavior of  $J(\nu, x)$  and  $Y(\nu, x)$  for  $r \rightarrow \infty$ , given by [58]

$$J(\nu, x) \sim \sqrt{\frac{2}{\pi x}} \cos \left( x - \frac{\pi \nu}{2} - \frac{\pi}{4} \right), \quad (83)$$

and

$$Y(\nu, x) \sim \sqrt{\frac{2}{\pi x}} \sin \left( x - \frac{\pi \nu}{2} - \frac{\pi}{4} \right), \quad (84)$$

respectively, we obtain

$$f_{|j|}(r) \sim C_j \sqrt{\frac{2}{\pi k r}} \left[ \cos \left( k r - \frac{\pi |j|}{2} - \frac{\pi}{4} \right) - \mu_j^{\lambda_j^s} \sin \left( k r - \frac{\pi |j|}{2} - \frac{\pi}{4} \right) \right]. \quad (85)$$

By comparing the above expression with Eq. (82), we have

$$\cos \left( k r - \frac{\pi |j|}{2} - \frac{\pi}{4} + \theta_{\lambda_j^s} \right) = \cos \left( k r - \frac{\pi |m|}{2} - \frac{\pi}{4} + \delta_j^{\lambda_j^s}(k, \bar{\phi}) \right), \quad (86)$$

with  $\theta_{\lambda_j^s}$  given as

$$\cos \theta_{\lambda_j^s} = C_j, \quad \sin \theta_{\lambda_j^s} = C_j \mu_j^{\lambda_j^s}. \quad (87)$$

Therefore, Eq. (82) is satisfied if

$$C_j = \left[ 1 + \left( \mu_j^{\lambda_j^s} \right)^2 \right]^{-1/2}. \quad (88)$$

Now, comparing the arguments of the cosines above, the following phase shift is achieved:

$$\delta_j^{\lambda_j^s}(k, \bar{\phi}) = \Delta_m(\bar{\phi}) + \theta_{\lambda_j^s}, \quad (89)$$

where

$$\Delta_m(\bar{\phi}) = \frac{\pi}{2} (|m| - |m + \bar{\phi}|), \quad (90)$$

and

$$\theta_{\lambda_j^s} = \arctan(\mu_j^{\lambda_j^s}). \quad (91)$$

Therefore, the scattering operator  $S_{\bar{\phi}, j}^{\lambda_j^s}$  ( $S$  matrix) for the self-adjoint extension is

$$S_{\bar{\phi}, j}^{\lambda_j^s} = e^{2i\delta_j^{\lambda_j^s}(k, \bar{\phi})} = e^{2i\Delta_m(\bar{\phi})} e^{2i\theta_{\lambda_j^s}}, \quad (92)$$

that is,

$$S_{\bar{\phi}, j}^{\lambda_j^s} = e^{2i\Delta_m(\bar{\phi})} \left[ \frac{1 + i\mu_j^{\lambda_j^s}}{1 - i\mu_j^{\lambda_j^s}} \right]. \quad (93)$$

Using Eq. (81), we have

$$S_{\bar{\phi}, j}^{\lambda_j^s} = e^{2i\Delta_m(\bar{\phi})} \left[ \frac{\lambda_j^s k^{2|j|} \Gamma(1 - |j|) e^{i|j|\pi} + 4^{|j|} \Gamma(1 + |j|)}{\lambda_j^s k^{2|j|} \Gamma(1 - |j|) e^{-i|j|\pi} + 4^{|j|} \Gamma(1 + |j|)} \right]. \quad (94)$$

In accordance with the general theory of scattering, the poles of the  $S$  matrix in the upper half of the complex plane [59] determine the positions of the bound states in the energy scale, Eq. (73). These poles occurs when the denominator of Eq. (94) is equal to zero and with the replacement  $k \rightarrow i\kappa$ . So, we have

$$\lambda_j^s(i\kappa)^{2|j|} \Gamma(1 - |j|) e^{-i|j|\pi} + 4^{|j|} \Gamma(1 + |j|) = 0. \quad (95)$$

Solving this equation for  $\mathcal{E}$ , we found

$$\mathcal{E} = -\frac{2}{M} \left[ -\frac{1}{\lambda_j^s} \frac{\Gamma(1 + |j|)}{\Gamma(1 - |j|)} \right]^{1/|j|}. \quad (96)$$

Thus, by comparing the expression above with Eq. (73), we find  $\lambda_j^s = \lambda_j$ , with  $\lambda_j$  given by Eq. (74), and the self-adjoint extension parameter for the scattering scenario being the same one as that for the bound state problem. This is a very interesting result first discussed in [34]. Thus, we also obtain the phase shift and the scattering matrix in terms of physics of the problem. If  $\lambda_j^s = 0$ , we achieve the corresponding result for the pure AB problem with Dirichlet boundary condition; in this case, we recover the expression for the scattering matrix found in Ref. [60],  $S_{\bar{\phi}, l}^{\lambda_j^s} = e^{2i\Delta_m(\bar{\phi})}$ . If we make  $\lambda_j^s = \infty$ , we get

$$S_{\bar{\phi}, l}^{\lambda_j^s} = e^{2i\Delta_m(\bar{\phi}) + 2i\pi|j|}. \quad (97)$$

The scattering amplitude  $f_{\bar{\phi}, \alpha}(k, \theta)$  can be obtained using the standard methods of scattering theory, namely ( $f_k = 1/\sqrt{2\pi i k}$ )

$$\begin{aligned} f_{\bar{\phi}, \alpha}(k, \theta) &= f_k \sum_{m=-\infty}^{\infty} \left( e^{2i\delta_j^{\lambda_j^s}(k, \bar{\phi})} - 1 \right) e^{im\theta} \\ &= f_k \sum_{m=-\infty}^{\infty} \left( e^{2i\Delta_m(\bar{\phi})} \left[ \frac{1 + i\mu_j^{\lambda_j^s}}{1 - i\mu_j^{\lambda_j^s}} \right] - 1 \right) e^{im\theta}. \end{aligned}$$



#### IV. AHARONOV-BOHM PROBLEM PLUS A TWO-DIMENSIONAL HARMONIC OSCILLATOR

In this section, we address the AB problem plus a two-dimensional harmonic oscillator (HO) located at the origin of a polar coordinate system. The potential of the harmonic oscillator in two-dimension space is given by

$$\hat{V}_{HO} = \frac{1}{2}M\omega_x^2x^2 + \frac{1}{2}M\omega_y^2y^2. \quad (98)$$

In polar coordinates  $(r, \varphi)$  it can be written as

$$\hat{V}_{HO} = \frac{1}{2}M\omega^2r^2, \quad (99)$$

where we are considering an isotropic HO,  $\omega_x = \omega_y = \omega$ .

Let us now include the oscillator potential (99) into the AB Hamiltonian (25), which leads to the following eigenvalue equation

$$\hat{H}_{HO}\Phi = i\partial_t\Phi, \quad (100)$$

where

$$\hat{H}_{HO} = \hat{H}_{NR} + \hat{V}_{HO}. \quad (101)$$

By writing the solutions of Eq. (100) in the form

$$\Phi(t, r, \varphi) = e^{-i\mathcal{E}'t} \begin{bmatrix} \phi_{\mathcal{E}'}(r)e^{i(m-s/2)\varphi} \\ \zeta_{\mathcal{E}'}(r)e^{i(m+s/2)\varphi} \end{bmatrix}, \quad (102)$$

we obtain the radial equation

$$\mathcal{H}'\phi_{\mathcal{E}'}(r) = \mathcal{E}'\phi_{\mathcal{E}'}(r), \quad (103)$$

where  $\mathcal{H}' = \mathcal{H}'_0 + \bar{\mathcal{U}}_{\text{short}}$ ,

$$\mathcal{H}'_0 = -\frac{1}{2M} \left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{j^2}{r^2} - \gamma^2 r^2 \right], \quad (104)$$

and  $\gamma^2 = M^2\omega^2$ .

In order to have a more detailed analysis of this problem, we will first examine the motion of the particle considering two cases (i) excluding the  $r = 0$  region and (ii) including the  $r = 0$  region afterwards. At the end, we compare with some results in the literature.

##### 3. Solution of the problem excluding $r = 0$ region

In this case, the Hamiltonian (101) does not contain  $\bar{\mathcal{U}}_{\text{short}}$ . By direct solving

$$\mathcal{H}'_0\phi_{\mathcal{E}'}(r) = \mathcal{E}'\phi_{\mathcal{E}'}(r). \quad (105)$$

we obtain [58]

$$\begin{aligned} \phi_{\mathcal{E}'}(r) &= A_j \gamma^{\frac{1+j}{2}} r^j e^{-\frac{1}{2}\gamma r^2} M(d, 1+j, \gamma r^2) \\ &+ B_j \gamma^{\frac{1+j}{2}} r^j e^{-\frac{1}{2}\gamma r^2} U(d, 1+j, \gamma r^2), \end{aligned} \quad (106)$$

where  $d = (1+j)/2 - M\mathcal{E}'/2\gamma$ ,  $U(a, b, z)$  is the confluent hypergeometric function of the second kind, and  $A_j, B_j$  are constants. However, as only  $M$  is regular at origin, it should be imposed  $B_j = 0$ . Moreover, if  $d$  is 0 or a negative integer, and  $1+|j|$  is a negative and non null integer, the series terminates and the hypergeometric function becomes a polynomial of degree  $n$  [58]. This condition guarantees that the hypergeometric function is regular at origin, which is essential for the treatment of the physical system since the region of interest is that around the charge filament. Therefore, the series in (106) must converge if we consider that  $d = -n$  and  $1+|j| \neq -\ell$ , where  $n, \ell \in \mathbb{Z}^*$ , with  $\mathbb{Z}^*$  denoting the set of the nonnegative integers. This condition also guarantees the normalizability of the wave function. So, using this condition, we obtain the discrete values for the energy whose expression is given by

$$\mathcal{E}' = (2n+1+|j|)\omega, \quad (107)$$

where  $n \in \mathbb{Z}$  and  $1+|j| \neq -\ell$ ,  $\ell \in \mathbb{Z}$ . The (unnormalized) energy wave function is given by

$$\begin{aligned} \Phi(r, \varphi) &= \gamma^{\frac{1+|j|}{2}} r^{|j|} e^{-\frac{1}{2}\gamma r^2} e^{i(m-s/2)\varphi} \\ &\times M(-n, 1+|j|, \gamma r^2), \end{aligned} \quad (108)$$

where  $C_j$  is a normalization constant. Notice that in Eq. (107)  $|j|$  can assume any non integer number larger than  $-1$ . However, we will see that this condition is no longer true when we include the term  $\bar{\mathcal{U}}_{\text{short}}$ . Next to study the motion of the particle in all space, including the  $r = 0$  region, we invoke the self-adjoint extension of the operator  $\mathcal{H}'_0$ .

##### 4. Solution including the $r = 0$ region

In this case, the dynamics includes the  $\bar{\mathcal{U}}_{\text{short}}$  term. So, let us follow the same procedure as in Sec. III A to find the bound states for the Hamiltonian  $\mathcal{H}'$ . Like before we need to find all the self-adjoint extension for the operator  $\mathcal{H}'_0$ . The relevant eigenvalue equation is

$$\mathcal{H}'_0\phi_{\vartheta, \mathcal{E}'}(r) = \mathcal{E}'\phi_{\vartheta, \mathcal{E}'}(r). \quad (109)$$

with  $\phi_{1, \vartheta}$  labeled by a parameter  $\vartheta$ . The solution to this equation are given in (106). However, the only square integrable functions is  $U(d, 1+j, \gamma r^2)$ . Then, this implies that  $A_j = 0$  in Eq. (106). Therefore, we get

$$\phi_{\vartheta, \mathcal{E}'}(r) = \gamma^{\frac{1+j}{2}} r^j e^{-\frac{1}{2}\gamma r^2} U(d, 1+j, \gamma r^2). \quad (110)$$

In order to guarantee that  $\phi_{\mathcal{E}'}(r) \in L^2(\mathbb{R}, r dr)$ , it is advisable to study their behavior as  $r \rightarrow 0$ , which implies analyzing the possible self-adjoint extensions.

Now, to construct the self-adjoint extensions, let us consider the eigenvalue equation

$$\mathcal{H}'_0^\dagger \phi_\pm(r) = \pm i k_0 \phi_\pm(r). \quad (111)$$

Since  $\mathcal{H}_0'^\dagger = \mathcal{H}_0'$  and, from (110), the solution to this equation is given by

$$\phi_\pm(r) = r^j e^{-\frac{\gamma r^2}{2}} U(d_\pm, 1 + j, \gamma r^2), \quad (112)$$

where  $d_\pm = (1 + j)/2 \mp ik_0/4\gamma$ .

Let us now consider the asymptotic behavior of  $U(d_\pm, 1 + j, \gamma r^2)$  as  $r \rightarrow 0$  [61]

$$U(d_\pm, 1 + j, \gamma r^2) \sim \left[ \frac{\Gamma(j)(\gamma r)^{-j}}{\Gamma(d_\pm)} + \frac{\Gamma(-j)r^j}{\Gamma(d_\pm - j)} \right]. \quad (113)$$

Working with this expression, let us find under which condition the term,

$$\int |\phi_{1,\pm}(r)|^2 r dr, \quad (114)$$

has a finite contribution near origin region. Taking Eqs. (112) and (113) into account, we have

$$\lim_{r \rightarrow 0} |\phi_{1,\pm}(r)|^2 r^{1+2j} \longrightarrow [A_1 r^{1+2j} + A_2 r^{1-2j}], \quad (115)$$

where  $A_1$  and  $A_2$  are constants. Studying Eq. (115), we see that  $\phi_\pm(r)$  is square-integrable only for  $j \in (-1, 1)$ . In this case, since  $\mathcal{N}_+$  is expanded by  $\phi_+(r)$  only, we have that its dimension is  $n_+ = 1$ . The same applies to  $\mathcal{N}_-$  and  $\phi_-(r)$ , resulting in  $n_- = 1$ . Then,  $\mathcal{H}_0'$  possesses self-adjoint extensions parametrized by a unitary matrix,  $\mathbb{U}(1) = e^{i\vartheta}$ , with  $\vartheta \in [0, 2\pi)$ . Therefore, the domain of  $\mathcal{H}_0^\dagger$  is given by

$$\mathcal{D}(\mathcal{H}_0'^\dagger) = \mathcal{D}(\mathcal{H}_0') \oplus \mathcal{N}_+ \oplus \mathcal{N}_-. \quad (116)$$

So, to extend the domain  $\mathcal{D}(\mathcal{H}_0')$  to match  $\mathcal{D}(\mathcal{H}_0'^\dagger)$  and therefore to turn  $\mathcal{H}_0'$  self-adjoint, we get

$$\mathcal{D}(\mathcal{H}_{0,\vartheta}') = \mathcal{D}(\mathcal{H}_0'^\dagger) = \mathcal{D}(\mathcal{H}_0') \oplus \mathcal{N}_+ \oplus \mathcal{N}_-, \quad (117)$$

we mean that, for each  $\vartheta$ , we have a possible domain for  $\mathcal{D}(\mathcal{H}_{0,\vartheta}')$ , but it will be the physical situation which will determine the value of  $\vartheta$ . The Hilbert space (117), for both cases of Eq. (42), contains vectors of the form

$$\begin{aligned} \phi_{\vartheta,\vartheta}(r) = \phi_{|j|}(r) + C \left\{ r^{|j|} e^{-\frac{\gamma r^2}{2}} \left[ U(d_+, 1 + |j|, \gamma r^2) \right. \right. \\ \left. \left. + e^{i\vartheta} U(d_-, 1 + |j|, \gamma r^2) \right] \right\}, \end{aligned} \quad (118)$$

where  $C$  is an arbitrary complex number,  $\phi_{|j|}(0) = \dot{\phi}_{|j|}(0) = 0$  and  $\phi_{|j|}(r) \in L^2(\mathbb{R}^+, r dr)$ . For a range of  $\vartheta$ , the behavior of the wave functions (118) was addressed in [62]. However, as we will see below, if we fix the physics of the problem at  $r = r_0$ , there is no need for such analysis because the value of  $\vartheta$  is automatically selected.

In order to find the spectrum of  $\mathcal{H}_0'$  we consider the limit  $r \rightarrow 0$  of  $\phi_{\vartheta,\vartheta}(r)$ , i.e., we substitute (110) in the left-hand side of (118) and, using (113), after equating the coefficients of the same power in  $r$  leads to

$$\frac{\mathcal{A}}{\Gamma(d)} = \mathcal{C} \left[ \frac{1}{\Gamma(d_+)} + \frac{e^{i\vartheta}}{\Gamma(d_-)} \right], \quad (119)$$

and

$$\frac{\mathcal{A}}{\Gamma(d - |j|)} = \mathcal{C} \left( \frac{1}{\Gamma(d_+ - |j|)} + \frac{e^{i\vartheta}}{\Gamma(d_- - |j|)} \right), \quad (120)$$

whose quotient leads to

$$\frac{\Gamma(d - |j|)}{\Gamma(d)} = \frac{\frac{1}{\Gamma(d_+)} + \frac{e^{i\vartheta}}{\Gamma(d_-)}}{\frac{1}{\Gamma(d_+ - |j|)} + \frac{e^{i\vartheta}}{\Gamma(d_- - |j|)}}. \quad (121)$$

The left-hand side of this equation is a function of the energy  $\mathcal{E}'$  while its right-hand side is a constant (even though it depends on the extension parameter  $\vartheta$  which is fixed by the physics of the problem). Then we have the equation

$$\frac{\Gamma\left(\frac{1-|j|}{2} - \frac{M\mathcal{E}'}{2\gamma}\right)}{\Gamma\left(\frac{1+|j|}{2} - \frac{M\mathcal{E}'}{2\gamma}\right)} = \text{const.} \quad (122)$$

Therefore, we have achieved the energy levels with an arbitrary parameter  $\vartheta$ , and we get inequivalent quantizations to different values of it [63]. Each physical problem selects a specific value of  $\vartheta$ .

Next we replace problem defined by the Hamiltonian  $\mathcal{H}' = \mathcal{H}_0' + \overline{\mathcal{U}}_{\text{short}}$ , by  $\mathcal{H}_0'$  plus self-adjoint extensions. So, we consider the static solutions ( $\mathcal{E}' = 0$ )  $\phi_0(r)$  and  $\phi_{\vartheta,0}(r)$  for the equations

$$[\mathcal{H}_0' + \overline{\mathcal{U}}_{\text{short}}] \phi_0(r) = 0, \quad (123)$$

$$\mathcal{H}_0' \phi_{\vartheta,0}(r) = 0. \quad (124)$$

Now, to find out the value of  $\vartheta$ , it is required the continuity of the logarithmic derivative

$$r_0 \frac{\dot{\phi}_0(r)}{\phi_0(r)} \Big|_{r=r_0} = r_0 \frac{\dot{\phi}_{\vartheta,0}(r)}{\phi_{\vartheta,0}(r)} \Big|_{r=r_0}. \quad (125)$$

The left-hand side of Eq. (125) is obtained by integration of Eq. (123) from 0 to  $r_0$ . Since the integration of the harmonic term

$$\gamma^2 \int_0^{r_0} r^{3+2|j|} \phi_0(r) dr \approx \gamma^2 \phi_0(r_0) \int_0^{r_0} r^{3+2|j|} dr \rightarrow 0, \quad (126)$$

as  $r_0 \rightarrow 0$ , the result is the same as in Eq. (52). Then, we arrive at

$$r_0 \frac{\dot{\phi}_0(r)}{\phi_0(r)} \Big|_{r=r_0} \approx \eta + \frac{j^2}{2}. \quad (127)$$

The right-hand side of Eq. (123) is carried out using the asymptotic behavior of  $U(d, 1 + |j|, \gamma r^2)$ , given in Eq. (113), in Eq. (118), and it take the form

$$r_0 \frac{\dot{\phi}_{\vartheta,0}(r)}{\phi_{\vartheta,0}(r)} \Big|_{r=r_0} = \frac{\dot{\Omega}'_{\vartheta}(r)}{\Omega'_{\vartheta}(r)} \Big|_{r=r_0}, \quad (128)$$

where

$$\Omega'_\vartheta(r) = \left[ \frac{\Gamma(|j|)(\gamma r)^{-|j|}}{\Gamma(d_+)} + \frac{\Gamma(-|j|)r^{|j|}}{\Gamma(d_+ - |j|)} \right] + e^{i\vartheta} \left[ \frac{\Gamma(|j|)(\gamma r)^{-|j|}}{\Gamma(d_-)} + \frac{\Gamma(-|j|)r^{|j|}}{\Gamma(d_- - |j|)} \right]. \quad (129)$$

Replacing Eqs. (127) and (128) in expression (125), we obtain

$$\frac{\dot{\Omega}'_\vartheta(r)}{\Omega'_\vartheta(r)} \Big|_{r=r_0} = \eta + \frac{j^2}{2}. \quad (130)$$

With this relation, we select an approximated value for parameter  $\vartheta$  in terms of the physics of the problem.

The solutions for (105) are given by (110) and since this function belongs to  $\mathcal{D}(H'_{0,\vartheta})$ , it is the form (118), that is,

$$\phi_{1,\vartheta}^{\mathcal{E}'}(r) = \phi_1(r) + C \left\{ r^{|j|} e^{-\frac{\gamma r^2}{2}} \left[ U(d_+, 1 + |j|, \gamma r^2) + e^{i\vartheta} U(d_-, 1 + |j|, \gamma r^2) \right] \right\}, \quad (131)$$

for some  $\vartheta$  selected from the physics of the problem at  $r = r_0$ . So, using (110) in (131) we have ( $\Theta = \Gamma(d)/\Gamma(d - |j|)$ )

$$\begin{aligned} r_0 \frac{\phi_{1,\vartheta}^{\mathcal{E}'}(r)}{\phi_{1,\vartheta}^{\mathcal{E}'}(r)} \Big|_{r=r_0} &= \frac{|j| \left[ r_0^{2|j|} \Gamma(-|j|) \gamma^{|j|} \Theta - \Gamma(|j|) \right]}{r_0^{2|j|} \Gamma(-|j|) \gamma^{|j|} \Theta + \Gamma(|j|)} \\ &= \frac{\dot{\Omega}'_\vartheta(r)}{\Omega'_\vartheta(r)} \Big|_{r=r_0}. \end{aligned} \quad (132)$$

Using the Eq. (130) in the above equation we obtain

$$\frac{\Gamma(d)}{\Gamma(d - |j|)} = -\frac{1}{\gamma^{|j|} r_0^{2|j|}} \left( \frac{j^2 - 2|j| + 2\eta}{j^2 + 2|j| + 2\eta} \right) \frac{\Gamma(1 + |j|)}{\Gamma(1 - |j|)}. \quad (133)$$

The Eq. (133) is too complicated to evaluate the bound state energy explicitly, but its limiting features are interesting. If we take limit  $r_0 \rightarrow 0$  in this expression, the bound state energy are determined by the poles of the gamma functions, i.e.,

$$\begin{aligned} -1 < j < 0, \quad \mathcal{E}' &= (2n + 1 - |j|)\omega, \\ 0 < j < 1, \quad \mathcal{E}' &= (2n + 1 + |j|)\omega, \end{aligned} \quad (134)$$

or

$$\mathcal{E}' = (2n + 1 \pm |j|)\omega, \quad (135)$$

where  $n \in \mathbb{Z}^*$ . The  $+$  ( $-$ ) sign refers to solutions which are regular (irregular) at the origin. This result coincide with the Eq. (1) of Ref. [64], for the special case of  $\alpha = 1$ . Another interesting case is that of vanishing harmonic oscillator potential. This is achieved using the asymptotic behavior of the ratio of gamma functions for  $\gamma \rightarrow 0$  [65],

$$\frac{\Gamma\left(\frac{1+|j|}{2} - \frac{M\mathcal{E}}{2\gamma}\right)}{\Gamma\left(\frac{1-|j|}{2} - \frac{M\mathcal{E}}{2\gamma}\right)} \sim \left( \frac{-M\mathcal{E}}{2\gamma} \right)^{|j|}, \quad (136)$$

which holds for  $\mathcal{E} < 0$  and this condition is necessary for the pure AB system to have a bound state. Using this limit in the Eq. (133), one finds

$$\left( \frac{-M\mathcal{E}}{2\gamma} \right)^{|j|} = -\frac{1}{\gamma^{|j|} r_0^{2|j|}} \left( \frac{j^2 - 2|j| + 2\eta}{j^2 + 2|j| + 2\eta} \right) \frac{\Gamma(1 + |j|)}{\Gamma(1 - |j|)}. \quad (137)$$

Then, by solving Eq. (137) for  $\mathcal{E}$ , we obtain

$$\mathcal{E} = -\frac{2}{Mr_0^2} \left[ \frac{\Gamma(1 + |j|)}{\Gamma(1 - |j|)} \left( \frac{j^2 - 2|j| + 2\eta}{j^2 + 2|j| + 2\eta} \right) \right]^{1/|j|}, \quad (138)$$

which is the result obtained in Eq. (61). Thus, in the limit of vanishing harmonic oscillator, we recover the pure AB problem.

Now we have to remark that this result contains a subtlety that must be interpreted as follows: the presence of the singularity in the problem establish the range of  $j$  as given by  $|j| < 1$ . If we ignore the singularity and impose that the wave function is regular at the origin ( $\phi(r) \equiv \dot{\phi}(r) \equiv 0$ ), we achieve the same spectrum of Eq. (135), but with  $j$  assuming any non integer value greater than  $-1$  [66–68]. In this sense the self-adjoint extension prevents us from obtaining a spectrum incompatible with the singular nature of the Hamiltonian when we have  $\overline{U}_{\text{short}}$  [69, 70]. We have to take into account that the true boundary condition is that the wave function must be square-integrable through all space, regardless it is irregular or regular at the origin [35, 70].

##### 5. Determination of self-adjoint extension parameter

For our intent, it is more convenient to write the solutions for (103) for  $r \neq 0$ , taking into account both cases in (42) simultaneously, as

$$\begin{aligned} \phi_{\mathcal{E}'}(r) &= A_j \gamma^{\frac{1+|j|}{2}} e^{-\frac{\gamma r^2}{2}} r^{|j|} M(d, 1 + |j|, \gamma r^2) \\ &\quad + B_j \gamma^{\frac{1-|j|}{2}} e^{-\frac{\gamma r^2}{2}} r^{-|j|} M(d - |j|, 1 - |j|, \gamma r^2), \end{aligned} \quad (139)$$

where  $A_j, B_j$  are the coefficients of the regular and singular solutions, respectively. By implementing Eq. (139) into the boundary condition (62), we derive the following relation between the coefficients  $A_j$  and  $B_j$ :

$$\lambda'_j A_j \gamma^{|j|} = B_j \left( 1 - \frac{\lambda'_j \mathcal{E}'}{4(1 - |j|)} \lim_{r \rightarrow 0^+} r^{2-2|j|} \right), \quad (140)$$

where  $\lambda'_j$  is the self-adjoint extension parameter for the AB plus HO system. In the above equation, the coefficient of  $B_j$  diverges as  $\lim_{r \rightarrow 0^+} r^{2-2|j|}$ , if  $|j| > 1$ . Thus,  $B_j$  must be zero for  $|j| > 1$ , and the condition for the occurrence of a singular solution is  $|j| < 1$ . So, the presence of an irregular solution stems from the fact the operator is not self-adjoint for  $|j| < 1$ , recasting the condition of non-self-adjointness of the previus sections, and this

irregular solution is associated with a self-adjoint extension of the operator  $\mathcal{H}'_0$  [55, 56]. In other words, the self-adjoint extension essentially consists in including irregular solutions in  $\mathcal{D}(\mathcal{H}'_0)$  to match  $\mathcal{D}(\mathcal{H}_0^\dagger)$ , which allows us to select an appropriate boundary condition for the problem.

In order to Eq. (139) to be a bound state,  $\phi_{j,\mathcal{E}'}(r)$  must vanish for large values of  $r$ , i.e., must be normalizable at large  $r$ . By using the asymptotic representation of  $M(a, b, z)$  for  $z \rightarrow \infty$ ,

$$M(a, b, z) \sim \frac{\Gamma(b)}{\Gamma(a)} e^z z^{a-b} + \frac{\Gamma(b)}{\Gamma(b-a)} (-z)^{-a}, \quad (141)$$

the normalizability condition yields the relation

$$B_j = -\frac{\Gamma(1+|j|)}{\Gamma(1-|j|)} \frac{\Gamma\left(\frac{1+|j|}{2} - \frac{M\mathcal{E}'}{2\gamma}\right)}{\Gamma\left(\frac{1-|j|}{2} - \frac{M\mathcal{E}'}{2\gamma}\right)} A_j. \quad (142)$$

From Eq. (140), for  $|j| < 1$  we have  $B_j = \lambda'_j \gamma^{|j|} A_j$  and by using Eq. (142), the bound state energy is implicitly determined by the equation

$$\frac{\Gamma\left(\frac{1+|j|}{2} - \frac{M\mathcal{E}'}{2\gamma}\right)}{\Gamma\left(\frac{1-|j|}{2} - \frac{M\mathcal{E}'}{2\gamma}\right)} = -\frac{1}{\lambda'_j \gamma^{|j|}} \frac{\Gamma(1+|j|)}{\Gamma(1-|j|)}. \quad (143)$$

This result coincides with Eq. (53) of Ref. [21] for  $\alpha = 1$ , as it should be. By comparing Eq. (143) with Eq. (133), we find

$$\frac{1}{\lambda'_j} = \frac{2}{r_0^{2|j|}} \left( \frac{j^2 - 2|j| + 2\eta}{j^2 + 2|j| + 2\eta} \right). \quad (144)$$

We have thus attained a relation between the self-adjoint extension parameter and the physical parameters of the problem.

The limiting features of Eq. (143) are interesting. For  $\lambda_j \rightarrow 0$  or  $\infty$ , from the poles of gamma function, we have

$$\begin{aligned} \lambda'_j &= 0, & \mathcal{E}' &= (2n+1+|j|)\omega, \\ \lambda'_j &= \infty, & \mathcal{E}' &= (2n+1-|j|)\omega. \end{aligned} \quad (145)$$

These bound states energies coincide with those regular and irregular solutions given in Eq. (135) of the previous section. From relation (144) the regular wave function, when  $\lambda'_j = 0$  and the spin is absence, it is associated with  $0 < j < 1$ , and the irregular wave function, when  $\lambda'_j = \infty$ , is associated with  $-1 < j < 0$ . Another interesting limit is that of the vanishing oscillator potential. So, using the limit (115) in (73), we have

$$\left( \frac{-M\mathcal{E}}{2\gamma} \right)^{|j|} = -\frac{1}{\lambda_j \gamma^{|j|}} \frac{\Gamma(1+|j|)}{\Gamma(1-|j|)}. \quad (146)$$

with  $\lambda_j$  the self-adjoint extension for the pure AB system. From this equation we have the spectrum for the pure AB system in terms of the self-adjoint extension parameter,

$$\mathcal{E} = -\frac{2}{M} \left[ -\frac{1}{\lambda_j} \frac{\Gamma(1+|j|)}{\Gamma(1-|j|)} \right]^{1/|j|}. \quad (147)$$

This result also coincides with Eq. (3.13) of Ref. [20] for  $\alpha = 1$ . By comparing this equation with (138) we arrive at

$$\frac{1}{\lambda_j} = -\frac{2}{r_0^{2|j|}} \left( \frac{j^2 - 2|j| + 2\eta}{j^2 + 2|j| + 2\eta} \right), \quad (148)$$

for the relation of the self-adjoint extension parameter and the physics of the problem for pure AB system. Then, the relation between the self-adjoint extension parameter and the physics of the problem for the pure AB has the same mathematical structure as for the AB plus HO. However, we must observe that the self-adjoint extension parameter is negative for the pure AB, confirming the restriction of negative values of the self-adjoint extension parameter made in [21], in such way we have an attractive  $\delta$  function. It is a necessary condition to have a bound state in the pure AB system.

## V. CONCLUSIONS

We have presented the detailed calculations of Ref. [34], in which was proposed a general regularization method to address a system endowed with a singular Hamiltonian (due to localized fields sources or quantum confinement). Using the KS approach, the bound states were determined in terms of the physics of the problem, in a very consistent way and without any arbitrary parameter. In the sequel, we employed the BG approach. By comparing the results of these approaches, we have determined an expression for the self-adjoint extension parameter for the bound state problem, which coincides with the one for scattering problem. We have thus obtained the S matrix in terms of the physics of the problem as well. In this point, we remark that the important results of Refs. [3, 20, 71] are giving in terms of the an arbitrary self-adjoint extension parameter. In our work this parameter was determined in terms of the physics of the problem. The outcomes obtained by Park are a particular case of our results for a fixed value of the self-adjoint extension parameter. To our knowledge, it was not known in the literature an expression for the bound state energies for the AB with a defined self-adjoint extension parameter. In Ref. [34] this expression was presented by the first time, whose details are derived here.

To illustrate the applicability of our approach to other physical systems, we deal with the spin-1/2 AB problem plus a two-dimensional harmonic oscillator. Two cases were considered: (i) without and (ii) with the inclusion of short range potential  $\overline{U}_{\text{short}}$  in the nonrelativistic Hamiltonian. Even though we have obtained an equivalent mathematical expression for both cases, has been shown that, in (i)  $j$  can assume any noninteger value greater than  $-1$  while in (ii) it is in the range  $j \in (-1, 1)$ . In the first case, it is reasonable to impose that the wave function vanish at the origin. However, this condition does not give a correct description of the problem in the  $r = 0$  region. Therefore, the energy spectrum obtained in the

second case is that physically acceptable. The presence of the singularity establishes that the effective angular momentum must obey the condition  $j \in (-1, 1)$  and implies that irregular solutions must be taken into account in this range. The only situation in which we can neglect the short range potential is the one in which one looks only for topological phases.

A natural extension of this work is the inclusion of the Coulomb potential [72], which naturally appears in two-dimensional systems, such as graphene [73] and anyons systems [74, 75].

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